Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 521–529
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Extended Helmholtz–Weyl decompositions

Niko Sauer*

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa

Received 13 September 2005; accepted 27 February 2006

Abstract

The Helmholtz–Weyl decomposition in which a vector field is decomposed into the curl of a vector potential and the gradient of a scalar potential, is extended to situations where function-objects somewhat different from vector fields are considered. This is done by creating, in the spirit of Hermann Weyl, a Hilbert space framework from which the classical as well as some new decompositions can be obtained. Because of the Hilbert space setting, functions in classes of square integrable functions are in the background. In applications to hydrodynamics, decomposition of the velocity field has to be brought into line with the decomposition of the time derivative of this field. For this purpose we introduce the *derived decomposition* which is what is really used in fluid mechanics. In addition there is the matter of dimensional correctness of decompositions to which attention is also paid. Applications of the theory to problems in fluid mechanics which involve *dynamic boundary conditions* are also given.

Keywords: Helmholtz–Weyl decomposition; Fluid mechanics; Dynamic boundary conditions

1. Introduction

The Helmholtz–Weyl decomposition [1,2] plays an important role in the mathematical study of incompressible fluid motion; most certainly in the study of the Navier–Stokes equations. See, e.g., [3]. It has an interesting history as recorded in [4, p. 102 ff.]. In the study of dynamic boundary conditions for incompressible fluid motions the traditional decomposition is no longer appropriate, since the (dynamic) boundary condition contains a pressure term [5–7]. In [5] the problem was addressed by constructing a projection related to the Helmholtz–Weyl decomposition in a product space, while in [7] the work lives with the boundary pressure term. In [6], where boundary permeations are studied, there is an underlying decomposition which is not explicitly identified.

In this paper, which is a full rendering (plus some expansions) of a conference proceedings article [8], a Hilbert space framework for the construction of Helmholtz–Weyl decompositions is established. It only applies to L^2-situations. For the decomposition of L^p-functions, a similar framework could be based on [9] and would be much more subtle.

* Corresponding author. Tel.: +27 12 420 3558; fax: +27 12 420 3893.
E-mail address: nikos@friedrichs.up.ac.za.

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.02.038

2. Bilinear forms in Hilbert space

We recapitulate a few aspects of the theory of bilinear forms in Hilbert space. A complete rendering may be found in [10] (see also [11,12]). We shall present the theory for real spaces. The results adapt without much effort to the complex case which is needed when holomorphic semigroups are considered [3]. The Hilbert space will be denoted by H, its inner product by (,) and the associated norm by $\| \|$.

The bilinear forms we consider will all be bounded and symmetric. They will be denoted by the symbols b, k and p. The quadratic form associated with, for example, b will be denoted by $\hat{b}$, i.e. $\hat{b}(u) = b(u, u)$. For complex spaces it is necessary to replace 'symmetric' by 'Hermitian' and the quadratic forms will be real-valued. Some crucial properties of bilinear forms are defined in terms of properties of the associated quadratic forms.

A bilinear form b is *nonnegative* if $\hat{b}(u) \geq 0$ for every $u \in H$. p is *positive definite* if there is a positive constant c such that $\hat{p}(u) \geq c\|u\|^2$ for all $u \in H$. k is *weakly continuous* if $u_n \rightharpoonup u$ (weak convergence) implies that $\hat{k}(u_n) \to \hat{k}(u)$. The bilinear form b is *elliptic* (*Legendre* in [10]) if it can be written in the form $b(u, v) = p(u, v) + k(u, v)$ with p positive definite and k weakly continuous. The *null-space* of a bilinear form b is defined as $N_b = \{u \in H : b(u, v) = 0 \text{ for all } v \in H\}$.

One of the most fundamental issues in the theory of bilinear forms is the solvability of the *functional equation*:

$$b(\varphi, q) = \langle \varphi, f \rangle \quad \text{for all } \varphi \in H, \tag{1}$$

with f a given bounded linear functional on H ($f \in H^*$). An important result in this regard is [10].

Theorem 1. *If b is elliptic the functional equation (1) is solvable if and only if $N_b \subset \ker f$. Solutions are unique up to addition of elements of N_b. The null-space N_b is finite dimensional.*

3. Decompositions induced by elliptic forms

Helmholtz–Weyl decompositions essentially involve two spaces. Let H_0 and H_1 be two complex Hilbert spaces. The inner products and norms will be denoted by corresponding subscripts. Let $M : H_1 \to H_0$ be a bounded linear operator, and define the bilinear form b on $H_1 \times H_1$ by

$$b(\varphi, q) := (M\varphi, Mq)_0 \quad \text{for } \varphi, q \in H_1. \tag{2}$$

The form b as defined in (2) is obviously symmetric and nonnegative. Moreover $N_b = \ker M$. We assume that b is elliptic. That is, a weakly continuous bilinear form k can be found such that $b - k$ is positive definite. Let us consider the functional equation

$$b(\varphi, q) = (M\varphi, Mu)_0 = (M\varphi, F)_0 \quad \textit{for all } \varphi \in H_1, \tag{3}$$

with F a given element of H_0. The mapping $\varphi \mapsto (M\varphi, F)_0$ is obviously in H^* and the kernel of this mapping contains N_b. Hence, by Theorem 1, Eq. (3) is solvable. Additionally it is seen from (3) that if q is a solution, Mq is unique. We may therefore define the linear operator $P : H_0 \to H_0$ by $PF = Mq$. Again, from (3) and the Schwarz inequality, it is seen that P is bounded. In fact, $\|PF\|_0 \leq \|F\|_0$. By the same uniqueness argument as above it is seen that $P^2 = P$. Therefore P is a projection.

We have to describe the projection $P^{\perp} := I - P$ orthogonal to P. For this purpose we re-write (3) in the form

$$b(\varphi, q) = (M\varphi, Mq)_0 = (\varphi, M^* F)_1 \quad \text{for all } \varphi \in H_1, \tag{4}$$

for $F \in \mathfrak{D}(M^*)$, and M^* the adjoint of M as defined in terms of the dual of M and the Riesz-map (see, e.g., [13, VII.2, p. 196]). Clearly, for $F \in \ker M^*$, $Mq = 0$, and hence $PF = 0$. Conversely, if $PF = 0$ and $F \in \mathfrak{D}(M^*)$, it follows from (4) that $F \in \ker M^*$.

We therefore have the following general decomposition result:

Theorem 2. *Suppose that the bilinear form b as defined in (2) is elliptic. If $F \in \mathfrak{D}(M^*)$ then F has the unique orthogonal decomposition*

$$F = Mq + w, \tag{5}$$

in H_0, with q a solution of (3) and $w \in \ker M^*$.

4. Derived decompositions

In applications, especially to problems in hydrodynamics, the function to be decomposed is often a time-derivative. We therefore need to study the relation between the decomposition of a derivative and the derivative of the decomposition. Let the setting be the same as before and consider functions $v : t \in (a, b) \mapsto v(t) \in H_0$, with (a, b) an open interval of the real line. Suppose that the function v has a derivative $\dot{v}(t)$ in H_0 at every point $t \in (a, b)$. That is, $\lim_{h\to 0} \|\dot{v}(t) - h^{-1}[v(t + h) - v(t)]\|_0 = 0$. Then the projections $Pv(t) = Mq(t)$ and $P\dot{v}(t) = Mq'(t)$ constructed in the previous section exist.

Theorem 3. *The function $t \in (a, b) \mapsto Mq(t)$ is differentiable in H_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}t} Mq(t) = Mq'(t) \quad \text{for all } t \in (a, b), \tag{6}$$

for some $q' \in H_1$. If $v(t) \in \mathfrak{D}(M^*)$ for all t, and has the orthogonal decomposition $v(t) = Mq(t) + w(t)$, then w is differentiable and

$$\dot{v}(t) = Mq'(t) + \dot{w}(t). \tag{7}$$

Furthermore $\dot{v}(t) \in \mathfrak{D}(M^*)$ and $\dot{w}(t) \in \ker M^*$. The decomposition (7) is orthogonal.

Proof. Let the difference quotient δ_h be defined by $\delta_h f(t) = h^{-1}[f(t + h) - f(t)]$. Then, by (3),

$$\left.\begin{aligned} (M\varphi, \delta_h Mq(t))_0 &= (M\varphi, \delta_h v(t))_0 \\ (M\varphi, Mq'(t))_0 &= (M\varphi, \dot{v}(t))_0. \end{aligned}\right\} \tag{8}$$

Subtraction of the two equations in (8) and setting $\varphi = \delta_h Mq(t) - Mq'(t)$ leads to the identity

$$\|\delta_h Mq(t) - Mq'(t)\|_0^2 = (\delta_h Mq(t) - Mq'(t), \delta_h v(t) - \dot{v}(t))_0. \tag{9}$$

Application of the Schwarz inequality in (9) yields the required result about the differentiability of $Mq(t)$ and the associated expression (7). The differentiability of $w(t)$ now follows directly. The placement of $\dot{v}$ and $\dot{w}$ follows from the fact that the adjoint operator M^* is closed. Orthogonality of the decomposition is clear. □

The expression (7) will be called the *derived decomposition*. In applications to physics and engineering, especially those concerned with incompressible fluid flows, the underlying evolution equation is of the form

$$\dot{v}(t) + Mq'(t) = F(v) \tag{10}$$

under the constraint

$$M^* v(t) = 0. \tag{11}$$

Equations such as (10) are formulated within the framework of the derived composition, while the constraint (11) is in the framework of the decomposition itself. Theorem 3 is then used to eliminate the 'pressure' term Mq' by taking the projection $P^\perp$ throughout to obtain

$$\dot{v}(t) = P^\perp F(v(t)). \tag{12}$$

Energy methods for the study of equations of the form (10) are often based on the taking of the inner product (in H_0) with $v(t)$. It is then found that, due to the constraint (11), the term Mq' miraculously disappears. The reason for this can be seen from the following calculation:

$$\begin{aligned} (\dot{v}(t), v(t))_0 + (Mq', P^\perp v(t))_0 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2 + (P^\perp Mq', v(t))_0 \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2 = (F(v), v(t))_0. \end{aligned} \tag{13}$$

These remarks will become clearer as we go on.

5. The traditional Helmholtz–Weyl decomposition

In this and the following sections Ω will denote a bounded open set in $R^n; n > 1$ with boundary Γ. It will be assumed that Γ is sufficiently smooth so that the natural embedding of the Sobolev space $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (Rellich's theorem holds). Throughout, we shall make the choice $H_1 = H^1(\Omega)$. This is a space of scalar fields. Vector fields and spaces of vector fields will be indicated in boldface letters.

To begin with we choose $H_0 = \boldsymbol{L}^2(\Omega)$, the space of square integrable n-vector fields defined on Ω. For $q \in H_1$ let $Mq := \nabla v \in H_0$. Then $b(\varphi, q) = (\nabla\varphi, \nabla q)_0$. If $J : v \in H_1 \mapsto v \in L^2(\Omega)$ is the (compact) natural embedding operator, the bilinear form $k(\varphi, q) := (J\varphi, Jq)_{L^2(\Omega)}$ is weakly continuous. Thus $b(\varphi, q) = (\varphi, q)_{H^1(\Omega)} - k(\varphi, q)$ is elliptic and Theorem 2 applies. This gives the traditional Helmholtz–Weyl decomposition. Indeed, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ for which $\gamma_0\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$, then $(M\varphi, \boldsymbol{v})_{L^2(\Omega)} = -(\varphi, \nabla \cdot \boldsymbol{v})_{H^1(\Omega)}$ as a simple integration by parts will show. Hence, such $\boldsymbol{v} \in \mathfrak{D}(M^*)$ and $M^*\boldsymbol{v} = \nabla \cdot \boldsymbol{v}$. Here the operator γ_0 is the Lions boundary trace and $\boldsymbol{\nu}$ the unit normal to the surface Γ.

6. A somewhat less conventional decomposition

Realizations of Theorem 2 are established by the choice of the spaces H_1, H_0 and the operator M. We have stated the choice of the space H_1 in the previous section. Let $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ and $\eta \in L^2(\Gamma)$ be given, not necessarily related, functions. Note that $\boldsymbol{v}$ is an n-vector field and η a scalar field. Let $H_0 := \boldsymbol{L}^2(\Omega) \times L^2(\Gamma)$ so that $\langle \boldsymbol{v}, \eta\rangle \in H_0$. We define the operator M by $Mq := \langle \nabla q, \gamma_0 q\rangle$. From (2) it is seen that, in this case,

$$b(\varphi, q) = (\nabla\varphi, \nabla q)_{L^2(\Omega)} + (\gamma_0\varphi, \gamma_0 q)_{L^2(\Gamma)}, \tag{14}$$

and the functional equation (3) takes the form

$$(\nabla\varphi, \nabla q)_{L^2(\Omega)} + (\gamma_0\varphi, \gamma_0 q)_{L^2(\Gamma)} = (\nabla\varphi, \boldsymbol{v})_{L^2(\Omega)} + (\gamma_0\varphi, \eta)_{L^2(\Gamma)}. \tag{15}$$

From (14) it is seen that the bilinear form b is positive. Indeed,

$$\hat{b}(\varphi) = \|\nabla\varphi\|^2_{L^2(\Omega)} + \|\gamma_0\varphi\|^2_{L^2(\Gamma)} \tag{16}$$

so that $\hat{b}(\varphi) = 0$ if and only if $\varphi = 0$. This shows that $\ker M$ is trivial. In addition b is elliptic as may be seen from (16). It follows, in fact, that $\hat{b}(\varphi) \geq \|\varphi\|^2_{H^1(\Omega)} - \|\varphi\|^2_{L^2(\Omega)}$ and the right of the inequality is elliptic by the Rellich embedding theorem. This implies that b itself is elliptic [10, Thm. 11.6, Cor. 2]. Since positive elliptic forms are positive definite [10, Thm. 11.1], it follows that b is positive definite. Thus the functional equation (15) has a unique solution q, and $P\langle \boldsymbol{v}, \eta\rangle = \langle \nabla q, \gamma_0 q\rangle$.

Formal integration by parts of (15) gives

$$-(\varphi, \Delta q)_{L^2(\Omega)} + (\gamma_0\varphi, \gamma_1 q + \gamma_0 q)_{L^2(\Gamma)} = -(\varphi, \nabla \cdot \boldsymbol{v})_{L^2(\Omega)} + (\gamma_0\varphi, \gamma_0\boldsymbol{v} \cdot \boldsymbol{\nu} + \eta)_{L^2(\Gamma)}. \tag{17}$$

The freedom of choice of φ in (17) enables us to conclude that q is a weak solution of the problem

$$\left.\begin{aligned} \Delta q &= \nabla \cdot \boldsymbol{v} \\ \gamma_1 q + \gamma_0 q &= \boldsymbol{v} \cdot \boldsymbol{\nu} + \eta, \end{aligned}\right\} \tag{18}$$

where the trace operator γ_1 denotes the normal derivative in the direction of $\boldsymbol{\nu}$.

It is seen from standard regularity theory for the elliptic boundary condition problem (18) that *if $\boldsymbol{v} \in \boldsymbol{H}^k(\Omega)$, $\eta \in H^{k-1/2}(\Gamma)$ for $k \geq 1$ and Γ is a k-times differentiable manifold, then $q \in H^{k+1}$ and $\boldsymbol{w} := \boldsymbol{v} - \nabla q \in \boldsymbol{H}^k$.*

If we take $\langle \boldsymbol{v}, \eta\rangle \in \boldsymbol{H}^1(\Omega) \times H^{1/2}(\Gamma)$, an inspection of (17) will show that $P\langle \boldsymbol{w}, \xi\rangle = 0$ if and only if $\nabla \cdot \boldsymbol{w} = 0$ and $\xi = -\gamma_0\boldsymbol{w} \cdot \boldsymbol{\nu}$. Thus we have

Theorem 4. *If $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and $\eta \in H^{1/2}(\Gamma)$, there exist unique $q \in H^2(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v} = \nabla q + \boldsymbol{w}$, $\eta = \gamma_0 q - \gamma_0\boldsymbol{w} \cdot \boldsymbol{\nu}$ and $\nabla \cdot \boldsymbol{w} = 0$.*

7. The Helmholtz–Weyl decomposition and incompressible flow

In applications to problems concerning the physical world the mathematical discussion we have had so far has to be adapted in order to ensure that the results are dimensionally correct. For example, if the variable x has the dimension of length, the inner product which generates the norm in the Sobolev space $H^1(\Omega)$ has to be defined as $(f,g)_{H^1(\Omega)} := (f,g)^2_{L^2(\Omega)} + \theta^2(\nabla f, \nabla g)_{L^2(\Omega)}$ with θ a constant parameter which has the physical unit of length. This parameter is often a natural part of the problem under consideration. For this and subsequent sections we take $n = 3$.

In traditional problems concerning incompressible flows, the equations of motion and the incompressibility constraint may be formulated in the form

$$\left.\begin{array}{l}\rho \boldsymbol{v}_t + \nabla p = \nabla \cdot \boldsymbol{S}(\boldsymbol{v})\\ \nabla \cdot \boldsymbol{v} = 0,\end{array}\right\} \tag{19}$$

with $\boldsymbol{S}$ a suitable tensor. Application of the traditional decomposition of $\boldsymbol{v}$ in the sense that the projection $P^\perp := I - P$ acting on $\boldsymbol{v}$ cannot knock out the term ∇p because $\boldsymbol{v}$ and ∇p do not have the same physical units and therefore live in different spaces. We are also not dealing with $\boldsymbol{v}$, but with its time derivative $\boldsymbol{v}_t$. Indeed, the discussion in Section 4 shows that we should be dealing with the derived decomposition and the projections associated with it.

A way out of this dilemma is to re-write the first of the equations in (19) in the form

$$\rho^{1/2} \boldsymbol{v}_t + \rho^{-1/2} \nabla p = \rho^{-1/2} \nabla \cdot \boldsymbol{S}, \tag{20}$$

which still is dimensionally correct but has a suitable symmetry. To see this we construct the traditional decomposition differently by choosing $M = \rho^{-1/2}\nabla$. The decomposition of $\boldsymbol{v}(t) := \boldsymbol{v}(\cdot, t) \in \boldsymbol{H}^1(\Omega)$ is now of the form

$$\rho^{1/2} \boldsymbol{v}(t) = \rho^{1/2} \boldsymbol{w}(t) + \rho^{-1/2} \nabla q$$

and this is dimensionally correct.

If we interpret the time derivative of $\boldsymbol{v}(t)$ as the time derivative in the norm of $H_0 = \boldsymbol{L}^2(\Omega)$, we see from Theorem 3 that $\boldsymbol{v}_t$ has the decomposition

$$\rho^{1/2} \boldsymbol{v}_t = \rho^{1/2} \boldsymbol{w}_t + \rho^{-1/2} \nabla q'$$

which is in accordance with (20). Moreover if $\boldsymbol{v}(t) \in \boldsymbol{H}^1(\Omega)$, $\nabla \cdot \boldsymbol{v}(t) = 0$ and $\gamma_0 \boldsymbol{v} \cdot \boldsymbol{\nu} = 0$, then $\boldsymbol{w}_t = \boldsymbol{v}_t$.

We note that in this particular framework the functional equation (3) becomes

$$\rho^{-1} (\nabla\varphi, \nabla q)_{L^2(\Omega)} = (\rho^{-1/2}\nabla\varphi, \rho^{1/2})_{L^2(\Omega)} = (\nabla\varphi, \boldsymbol{v})_{L^2(\Omega)},$$

which we can summarily re-write in the form

$$(\nabla\varphi, \nabla q)_{L^2(\Omega)} = \rho(\nabla\varphi, \boldsymbol{v})_{L^2(\Omega)}. \tag{21}$$

The functional equation (21) defines weak solutions of the Neumann problem

$$\left.\begin{array}{ll}\Delta q = \rho \nabla \cdot \boldsymbol{v} & \text{in } \Omega\\ \gamma_1 q = \rho \gamma_0 \boldsymbol{v} \cdot \boldsymbol{\nu} & \text{on } \Gamma.\end{array}\right\} \tag{22}$$

8. Incompressible flows in regions with permeable boundaries

The decomposition constructed in Section 6 may not be so far-fetched. We examine a canister filled with incompressible fluid, immersed in fluid of the same kind. It is assumed that the wall of the canister admits normal flow through it. The question is whether normal stress effects due to the flow will cause fluid to move through the boundary. The interior of the canister is denoted by Ω and the permeable wall by Γ. Modelling of the situation has led to equations of the form (see [6]):

$$\left.\begin{array}{l}\rho^{1/2}\boldsymbol{v}_t + \rho^{-1/2}\nabla p = \rho^{-1/2}\nabla\cdot\boldsymbol{S}(\boldsymbol{v}) \quad \text{in } \Omega \\ \gamma_0\boldsymbol{v} = -\eta\boldsymbol{v} \quad \text{at } \Gamma \\ \sigma^{1/2}\eta_t + \sigma^{-1/2}\gamma_0 p = \sigma^{-1/2}S(\eta,\boldsymbol{v}) \quad \text{on } \Gamma.\end{array}\right\} \tag{23}$$

Here $\sigma(x)$ represents a surface density of fluid particles which is assumed to be bounded below and above by positive numbers. The dynamic boundary condition in (23) (third equation) models the effect of stress-induced forces at the boundary.

The projection we construct is designed to keep the pair $\langle\rho^{1/2}\boldsymbol{v},\sigma^{1/2}\eta\rangle$ intact and eliminate the 'pressure' couple $\langle\rho^{-1/2}\nabla p,\sigma^{-1/2}\gamma_0 p\rangle$. To this end we let $H_0 = \boldsymbol{L}^2(\Omega)\times L^2(\Gamma)$, and define the operator M by $Mq = N_p\langle\nabla q,\gamma_0 q\rangle$ where

$$N_p = \begin{pmatrix}\rho^{-1/2}I_3 & 0\\ 0 & \sigma^{-1/2}\end{pmatrix}.$$

We see that $\|Mq\|_0^2 = \rho^{-1}\|\nabla q\|^2_{L^2(\Omega)} + \|\sigma^{-1/2}\gamma_0 q\|^2_{L^2(\Gamma)}$. The ellipticity follows as before. However, it is readily seen that $\|Mq\|_2^2 = 0$ if and only if $q = 0$, which implies that $\ker M$ is trivial and that b is positive definite as is the case in Section 6. Hence, for given $\langle\boldsymbol{v},\eta\rangle \in H_0$, the functional equation

$$\begin{aligned}b(\varphi,q) &= \rho^{-1}(\nabla\varphi,\nabla q)_{L^2(\Omega)} + (\gamma_0\varphi,\sigma^{-1}\gamma_0 q)_{L^2(\Gamma)}\\ &= N_p\langle\nabla\varphi,\gamma_0\varphi\rangle, N_p^{-1}\langle\boldsymbol{v},\eta\rangle_0\\ &= (\nabla\varphi,\boldsymbol{v})_{L^2(\Omega)} + (\gamma_0\varphi,\eta)_{L^2(\Gamma)}\end{aligned}$$

has a unique solution q. We also see that $P\langle\rho^{1/2}\boldsymbol{v},\sigma^{1/2}\rangle = PN_p^{-1}\langle\boldsymbol{v},\eta\rangle = 0$ if and only if $(\nabla\varphi,\boldsymbol{v})_{L^2(\Omega)} + (\gamma_0\varphi,\eta)_{L^2(\Gamma)} = 0$ for arbitrary φ. If $\boldsymbol{v}$ is sufficiently regular, this is the case if and only if $\nabla\cdot\boldsymbol{v}$ and $\eta = -\boldsymbol{v}\cdot\gamma_0\boldsymbol{v}$.

The function q is the weak solution of the following problem:

$$\left.\begin{array}{l}\Delta q = \rho\nabla\cdot\boldsymbol{v} \quad \text{in } \Omega\\ \rho^{-1}\gamma_1 q + \sigma^{-1}(x)\gamma_0 q = \boldsymbol{v}\cdot\gamma_0\boldsymbol{v} + \eta \quad \text{on } \Gamma.\end{array}\right\} \tag{24}$$

The decomposition theorem derived from (24) is the following:

Theorem 5. *Let $\boldsymbol{v}_t \in \boldsymbol{H}^k(\Omega)$; $k \geq 1$ and $\eta_t \in H^{k-1/2}(\Gamma)$. Then there exists a unique $q \in H^{k+1}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}^k(\Omega)$ such that*

$$\begin{aligned}&\rho^{1/2}\boldsymbol{v} = \rho^{-1/2}\nabla q + \rho^{1/2}\boldsymbol{w}\\ &\sigma^{1/2}\eta = \sigma^{-1/2}\gamma_0 q - \sigma^{1/2}\boldsymbol{v}\cdot\gamma_0\boldsymbol{w}\\ &\nabla\cdot\boldsymbol{w} = 0.\end{aligned}$$

If $\nabla\cdot\boldsymbol{v} = 0$ and $\eta = -\gamma_0\boldsymbol{v}_t\cdot\boldsymbol{v}$, then $\boldsymbol{w} = \boldsymbol{v}$.

The procedure around the derived decomposition is analogous to that discussed in the previous section.

9. Rotational motion of rigid bodies in a fluid

In [5] existence and uniqueness results for the purely rotational motion of a rigid body about an axis of symmetry in a Navier–Stokes fluid were obtained with the aid of semigroup-like methods based on [14]. The equations of motion of the rigid body include torques due to drag effects at the boundary, and thus include pressure terms as part of a dynamic boundary condition. A modified Helmholtz–Weyl decomposition had to be constructed for the purpose. In this section we describe the decomposition as a special example of the theory developed in Sections 3 and 4.

Consider a container with fluid in which a rigid body rotates about an axis of symmetry. Let Γ_1 denote the boundary of the container and Γ_2 the boundary of the rigid body. The boundaries Γ_1 and Γ_2 are assumed to be smooth. The region occupied by the fluid is denoted by Ω. Our analysis will be only for the situation in which Γ_1 and Γ_2 are sufficiently separated so that the restricted cone property and therefore the Rellich embedding theorem holds. We shall choose the origin of the spatial reference system on the rotation axis.

Let $\boldsymbol{v}$ and p denote the velocity and pressure fields of the fluid, and $\boldsymbol{I}$ the moment of inertia tensor of the rigid body about the rotation axis which passes through the origin $\boldsymbol{c}$. Let $\boldsymbol{\omega}$ denote the angular velocity of the rigid body relative to the chosen frame of reference. We shall assume no-slip boundary conditions of the form $\gamma_0\boldsymbol{v} = \boldsymbol{0}$ on Γ_1 and $\gamma_0\boldsymbol{v} = \boldsymbol{x} \wedge \boldsymbol{\omega}$ on Γ_2. The wedge denotes the vector product.

The equations of motion concerned are written as follows:

$$\left.\begin{aligned} &\rho^{1/2}\boldsymbol{v}_t + \rho^{-1/2}\nabla p = \rho^{-1/2}\nabla\cdot\boldsymbol{S} \\ &\boldsymbol{I}^{1/2}\dot{\boldsymbol{\omega}} + \boldsymbol{I}^{-1/2}\mathscr{S}p = \boldsymbol{I}^{-1/2}\int_{\Gamma_2}\boldsymbol{x}\wedge\boldsymbol{S}\boldsymbol{v}\,\mathrm{d}s \\ &\nabla\cdot\boldsymbol{v} = 0, \end{aligned}\right\} \tag{25}$$

where ρ denotes the density of the fluid and the dot ordinary differentiation with respect to t. The operator $\mathscr{S}$ is defined by

$$\mathscr{S}p = \int_{\Gamma_2(t)}\gamma_0 p\boldsymbol{x}\wedge\boldsymbol{v}\,\mathrm{d}s \tag{26}$$

with $\boldsymbol{v}$ the unit exterior normal to the boundary surface Γ_2.

We wish to construct a projection which will leave the couple $\langle\rho^{1/2}\boldsymbol{v}, \boldsymbol{I}^{1/2}\boldsymbol{\omega}\rangle$ intact while annihilating the pressure couple $\langle\rho^{-1/2}\nabla p, \boldsymbol{I}^{-1/2}\mathscr{S}p\rangle$.

Direct calculations and some reflection show that the operator $\mathscr{S}$ maps constant functions to zero.

For the application of the results of Section 3, and led by the discussion in Section 6, we let $H_0 = \boldsymbol{L}^2(\Omega)\times R^3$ and define the operator M as follows: $Mq := \boldsymbol{N}_r\langle\nabla q, \mathscr{S}q\rangle$ where

$$N_r := \begin{pmatrix}\rho^{-1/2}I_3 & 0 \\ 0 & \boldsymbol{I}^{-1/2}\end{pmatrix},$$

and I_3 is the 3×3 identity matrix. The functional equation (3) in this case is

$$\begin{aligned} b(\varphi, q) &:= \rho^{-1}(\nabla\varphi, \nabla q)_{L^2(\Omega)} + \boldsymbol{I}^{-1}\mathscr{S}\varphi\cdot\mathscr{S}q \\ &= (M\varphi, N_r^{-1}\langle\boldsymbol{v}_t, \dot{\boldsymbol{\omega}}\rangle)_0 \\ &= (\langle\nabla\varphi, \mathscr{S}\varphi\rangle, \langle\boldsymbol{v}, \boldsymbol{\omega}\rangle)_0 \\ &= (\nabla\varphi, \boldsymbol{v})_{L^2(\Omega)} + \mathscr{S}\varphi\cdot\boldsymbol{\omega}. \end{aligned} \tag{27}$$

It follows that

$$\begin{aligned} \hat{b}(q) &= \|Mq\|_0^2 = \rho^{-1}\|\nabla q\|_{L^2(\Omega)}^2 + |\boldsymbol{I}^{-1/2}\mathscr{S}q|^2 \\ &\geq \rho^{-1}(\|q\|_{H^1(\Omega)}^2 - \|q\|_{L^2(\Omega)}^2), \end{aligned}$$

and hence the form b is elliptic. It is also seen that $\ker M$ consists only of constant functions. Theorem 2 therefore applies and we proceed to obtain the meaning of the decomposition. To begin with, we notice that for a vector $\boldsymbol{\xi}$ which does not depend on position and $\boldsymbol{r}(x) = \boldsymbol{x}$

$$\boldsymbol{\xi}\cdot\mathscr{S}\varphi = -(\gamma_0\varphi, [\boldsymbol{r}\wedge\boldsymbol{\xi}]\cdot\boldsymbol{v})_{L^2(\Gamma_2)}. \tag{28}$$

With this taken into account, a formal integration by parts shows that $P\langle\rho^{-1/2}\boldsymbol{w}, \boldsymbol{I}^{-1/2}\boldsymbol{\xi}\rangle = PN_r^{-1}\langle\boldsymbol{w}, \boldsymbol{\xi}\rangle = 0$ if and only if

$$-(\varphi, \nabla\cdot\boldsymbol{w})_{L^2(\Omega)} + (\gamma_0\varphi, [\gamma_0\boldsymbol{w} - \boldsymbol{r}\wedge\boldsymbol{\xi}]\cdot\boldsymbol{v})_{L^2(\Gamma_2)} + (\gamma_0\varphi, \gamma_0\boldsymbol{v}\cdot\boldsymbol{v})_{L^2(\Gamma_1)} = 0. \tag{29}$$

Once again the freedom of choice afforded by φ, allows us to translate the condition (29) to

$$\left.\begin{array}{l}\nabla\cdot\boldsymbol{w}=0\\ \gamma_0\boldsymbol{w}-\boldsymbol{r}\wedge\boldsymbol{\xi}\quad\text{is tangential to }\Gamma_2\\ \gamma_0\boldsymbol{w}\quad\text{is tangential to }\Gamma_1.\end{array}\right\}\tag{30}$$

These conclusions are, however, only formal and have to be supported by regularity of the functions involved. We note, once again by formal partial integration of (27) and taking cognisance of (28), that q is a weak solution of

$$\left.\begin{array}{l}\Delta q=\rho\nabla\cdot\boldsymbol{v}\\ \rho^{-1}\gamma_1 q-\boldsymbol{I}^{-1}\mathscr{S}q\cdot\boldsymbol{v}=[\gamma_0\boldsymbol{v}-\boldsymbol{r}\wedge\boldsymbol{\omega}]\cdot\boldsymbol{v}.\end{array}\right\}\tag{31}$$

Elliptic regularity theory gives the decomposition

Theorem 6. *Suppose that for $k\geq 1$, $\boldsymbol{v}\in\boldsymbol{H}^k(\Omega)$, then there exist $q\in H^k(\Omega)$ (unique up to the addition of a constant), $\boldsymbol{w}\in\boldsymbol{H}^k$ and $\boldsymbol{\xi}\in R^3$ such that*

$$\rho^{1/2}\boldsymbol{v}=\rho^{1/2}\boldsymbol{w}+\rho^{-1/2}\nabla q$$
$$\boldsymbol{I}^{1/2}\boldsymbol{\omega}=\boldsymbol{I}^{1/2}\boldsymbol{\xi}+\boldsymbol{I}^{-1/2}\mathscr{S}q,$$

and the conditions (30) are fulfilled. If the couple $\langle\boldsymbol{v},\boldsymbol{\omega}\rangle$ satisfies (30), then q is constant, $\boldsymbol{w}=\boldsymbol{v}$, and $\boldsymbol{\xi}=\boldsymbol{\omega}$.

10. Notes on energy identities

As in the case of incompressible viscous fluid motion under homogeneous Dirichlet boundary conditions, appropriate energy methods for the systems of dynamical equations (23) and (25) also have the effect of pressure terms disappearing.

In the case of the Eq. (23) the scalar product is taken with $\boldsymbol{N}_p^{-1}\langle\boldsymbol{v},\eta\rangle$ which results in the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho\|\boldsymbol{v}\|^2+\int_\Gamma\sigma(x)\eta^2(x,t)\mathrm{d}s\right]+(\boldsymbol{S},\boldsymbol{A})=0,$$

where $\boldsymbol{A}=\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T$.

Treatment of this identity within the context of stability of motion of fluids of the second grade requires additional constraints which can be found in [6].

For the Eq. (25) the energy method consists of taking the scalar product of the system of equations with $N_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle$. We notice that under the boundary condition imposed, $PN_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle=N_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle$ with the result that

$$\begin{aligned}(\boldsymbol{N}_r\langle\nabla p,\mathscr{S}p\rangle,\boldsymbol{N}_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle)&=(\boldsymbol{N}_r\langle\nabla p,\mathscr{S}p\rangle,P(t)\boldsymbol{N}_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle)\\&=(P(t)\boldsymbol{N}_r\langle\nabla p,\mathscr{S}p\rangle,\boldsymbol{N}_r^{-1}\langle\boldsymbol{v},\boldsymbol{\omega}\rangle)=0\end{aligned}$$

which shows why the pressure terms fall away. Integration by parts and some manipulation yields the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho\|\boldsymbol{v}\|^2+\boldsymbol{\omega}\cdot\boldsymbol{I}\boldsymbol{\omega}]+(\boldsymbol{S},\boldsymbol{A})=0.$$

References

[1] H. Helmholtz, Über die Theorie der Elektrodynamik. Erste Abhandlung. Über die Bewegungsgleichungen der Elektricität für ruhende leitende Körper, J. Reine Angew. Math. 72 (1870) 57–129.

[2] H. Weyl, The method of orthogonal projection in potential theory, Duke Math. J. 7 (1940) 411–444.

[3] H. Fujita, T. Kato, On the Navier–Stokes initial value problem. I, Arch. Ration. Mech. Anal. 16 (1964) 269–315.

[4] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Volume I. Linearised Steady Problems, revised edition, in: Springer Tracts in Natural Philosophy, vol. 38, Springer Verlag, New York, 1998.

[5] M. Grobbelaar-van Dalsen, N. Sauer, Dynamic boundary conditions for the Navier–Stokes equations, Proc. Roy. Soc. Edinburgh Sect. A 113 (1989) 1–11.

[6] R. Maritz, N. Sauer, On boundary permeation in Navier–Stokes and second grade incompressible fluids, Math. Models Methods Appl. Sci. 16 (2006) 59–76.
[7] H. Beirão da Veiga, On the existence of strong solutions to a coupled fluid-structure evolution problem, J. Math. Fluid Mech. 6 (2004) 21–52.
[8] N. Sauer, On Helmholtz projections, in: R. Salvi (Ed.), Navier–Stokes Equations: Theory and Numerical Analysis, in: Pitman Research Notes in Mathematics, vol. 388, 1998, pp. 257–263.
[9] N. Sauer, Riesz–Fréchet-type theorems in locally convex spaces, Math. Z. 93 (1967) 349–354.
[10] M.R. Hestenes, Applications of the theory of quadratic forms in Hilbert space to the calculus of variations, Pacific J. Math. 1 (1951) 525–581.
[11] S. Hildebrandt, Rand-und Eigenwertaufgaben bei stark elliptischen Systemen linearer Differentialgleichungen, Math. Ann. 148 (1962) 411–429.
[12] P.R. Halmos, Introduction to Hilbert Space and the Theory of Spectral Multiplicity, second edition, Chelsea Publishing Company, New York, 1957.
[13] K. Yosida, Functional Analysis, Springer Verlag, Berlin, Göttingen, Heidelberg, 1965.
[14] N. Sauer, Linear evolution equations in two Banach spaces, Proc. Roy. Soc. Edinburgh 91A (1982) 287–303.